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A Construction for Fitting Formations

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1. INTRODUCTION

A Fitting formation is a class of (finite soluble) groups that is both a Fitting class and a formation (in this paper all groups will be finite and soluble). While examples of saturated Fitting formations abound, and include many of the well known classes, examples of non-saturated Fitting formations are very meagre. The study of Fitting formations was begun by Hawkes [7] in 1970; in that paper he gave the first examples of non-saturated Fitting formations. Some time later Berger and Cossey [2] gave some more examples, using much the same ideas as Hawkes. Recently, in his Ph.D. thesis [12], one of us (C.L.K.) gave a unified treatment and generalization of the examples of Hawkes and of Berger and Cossey, and produced a new class of examples (these are described in Section 5 of this paper). Our aim in this paper is to show that all these examples arise as special cases of a more general construction.

We will show that the ideas implicit in Hawkes and in Berger and Cossey for constructing Fitting formations can be made explicit. Following Kanes [12], we suppose that for each group we have a class of modules defined, and that the family of these classes satisfies certain closure properties. We then construct a class of groups by using this family to place restrictions on the chief factors of the groups in the class. That the class so defined is a Fitting formation is a consequence of the closure properties of the family of classes of modules (Theorem 3.1). The problem then is to find appropriate classes of modules; the family we give in this paper is inspired

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by the class of characters, called π-factorable characters, introduced by Isaacs in [11]. Isaacs works with complex valued characters, while we will want to work with modules over fields of prime characteristic (not necessarily coprime to the orders of the groups involved). Because of this, and because we need some slight generalizations of his ideas, we give a development of the material from our point of view: we emphasize however that there are no new ideas involved and the only new result of this section is Theorem 2.7. It was the realization that the modules involved in Hawkes and in Berger and Cossey were π-special, and the modules involved in the new examples of Kanes were π-factorable, that lead us to the examples of the present paper.

Our main interest is in constructing non-saturated Fitting formations, and it is easy to see that the examples we construct are often, but not always, non-saturated. A complete description of the saturated formations among the examples of Kanes [12] has been given by Kovács [13]: a modification of his description works for our examples (Theorem 4.2), and we are grateful to him for permission to include the description here. Though the description is rather technical, the saturated formations among our examples are just the obvious ones.

Our notation is standard, and we assume familiarity with the basic ideas and definitions of classes of finite soluble groups (such as may be found in Gaschütz [6] for example), and of representation theory (we generally follow the usage of Curtis and Reiner [4] for modules, and Isaacs [10] for characters). We remind the reader that $\mathbf{S}$ denotes the class of all finite soluble groups, and, for a set of primes π, $\mathbf{S}_\pi$ the class of all finite soluble π-groups.

2. Factorable Modules

Throughout this section, K will denote an algebraically closed field of characteristic q, q a fixed prime, and π a set of primes.

We let $\mathrm{Irr}_K(G)$ denote the class of all irreducible KG-modules. If U is any KG-module, then each $g \in G$ determines a linear transformation of U: we will denote the determinant of this linear transformation by $\det(g \text{ on } U)$, and its multiplicative order (as a root of unity) by $o(g \text{ on } U)$.

2.1. Definition. Let $U \in \mathrm{Irr}_K(G)$. Then U is called π-special if

(i) $\dim U$ is a π-number, and

(ii) for S a subnormal of G, V an irreducible constituent of U_S, then if $S_{\pi'}$ is a Hall π'-subgroup of S, $\det(s \text{ on } V) = 1$ for each $s \in S_{\pi'}$.

Gajendragadkar [5] introduced the idea of π-special characters, and established their basic properties. The same π-special was introduced by Isaacs, who has considerably developed and refined the theory of π-special characters. Our definition of π-special modules is derived from their definition of π-special characters; the properties we want are proved by arguments similar to those of Isaacs and Gajendragadkar. There is a sufficient difference in our point of view and in the results we need to present the proofs rather than leaving it to the reader to translate. Throughout this section it would be preferable if the restriction to algebraically closed fields could be removed. The problem in extending the results to the non-algebraically closed case seems to be in understanding the composition factors of modules induced from irreducibles of normal subgroups, especially in the case where the irreducible is invariant in the whole group. Given enough restrictions on the set of primes involved, one can obtain results which can be applied to the main construction of this paper. However, at present the results are rather limited compared with the algebraically closed case. We will not explore this further here, but hope to return to the problem later.

The first result we need is almost trivial.

2.2. LEMMA. *Let $U \in \operatorname{Irr}_K(G)$, U π-special, and let S be a subnormal subgroup of G. If V is an irreducible constituent of U_S, then V is π-special.*

2.3. LEMMA. *Let N be a normal subgroup of G, and let $U \in \operatorname{Irr}_K(N)$, U π-special.*

(i) *If G/N is a π-group, then every composition factor of U^G is π-special.*

(ii) *If G/N is a π'-group, and U is invariant in G, then U^G has a unique π-special composition factor $\hat{U}$, and $\hat{U}_N = U$.*

Proof. (i) The proof is by induction on $|G|$, being clearly true for $|G| = 1$.

Suppose $S \lhd \lhd G$, $S \neq G$. Then the composition factors of $U_{S \cap N}$ are π-special by Lemma 2.2, and hence so are all composition factors of ${U_{S \cap N}}^S$ by the inductive hypothesis. That all composition factors of ${U^G}_S$ are π-special follows from the Mackey subgroup theorem. Now suppose that W is a composition factor of U^G: then $W_N = W_1 \oplus \cdots \oplus W_n$, where each W_i is isomorphic to a conjugate of U. Moreover, n divides $|G/N|$, and so dim W is a π-number; and if g is an element of π'-order of G, $g \in N$, and so $\det(g \text{ on } W) = \prod \det(g \text{ on } W_i) = 1$, since all the W_i are π-special.

(ii) It will be enough to prove the result for N maximal in G: the result will then follow by induction on $|G/N|$. Thus we assume N is a maximal normal subgroup of G.

It follows from Lemma 1 of Becker [1] that U has a unique extension to an irreducible KG-module $\hat{U}$ such that $\det(g \text{ on } \hat{U}) = 1$ for every element g of G of π'-order: following Becker, we call such an extension a 1-extension.

We now prove by induction on $|G:S|$ that if $S \lhd \lhd G$, and if $\hat{U}$ is the unique 1-extension of a π-special module U for some maximal normal subgroup N of π'-index, then for every irreducible constituent V of $\hat{U}_S$, and for every element g of S of π'-order, $\det(g \text{ on } V) = 1$.

Let M be a maximal normal subgroup of G containing S. If $M = N$, so that $S \leqslant N$, there is nothing to prove. Hence suppose that $M \neq N$, and let W be an irreducible constituent of $\hat{U}_M$, so that $\hat{U}_M = W^{x_1} \oplus \cdots \oplus W^{x_t}$ for some $x_1, ..., x_t$. Note that t is a π-number; and if H is a Hall π'-subgroup of M, $y \in H$, we have

$$\begin{aligned} 1 &= \det(y \text{ on } \hat{U}) \\ &= \prod_{i=1}^{t} \det(y \text{ on } W^{x_i}) \\ &= \prod_{i=1}^{t} \det(y^{x_i} \text{ on } W) \\ &= \det(y \text{ on } W)^t \prod_{i=1}^{t} \det([y, x_i] \text{ on } W). \end{aligned}$$

Since $MN = G$, we may assume that the x_i were chosen in N, and so $[y, x_i] \in M \cap N$. But all the irreducible constituents of $W_{M \cap N}$ are π-special, so that $o([y, x_i] \text{ on } W)$ is a π-number. Thus $\det(y \text{ on } W)^t$ (and also therefore $\det(y \text{ on } W)$) has multiplicative order a π-number. Since $o(y \text{ on } W)$ is a π'-number, we have

$$\det(y \text{ on } W) = 1.$$

Now M, W, $W_{M \cap N}$, $M \cap N$ satisfy the inductive hypotheses; so that if V is an irreducible constituent of W_S (and hence of $\hat{U}_S$), $\det(g \text{ on } V) = 1$ for every element g of π'-order in S. This completes the proof.

2.4. THEOREM. *Let U, $V \in \mathrm{Irr}_K(G)$ be π-special and π'-special respectively. Then $U \otimes V$ is irreducible. Moreover, if U', $V' \in \mathrm{Irr}_K(G)$ are π-special and π'-special, respectively, and $U \otimes V \cong U' \otimes V'$, then $U \cong U'$ and $V \cong V'$.*

Proof. The proof is by induction $|G|$: the result is clearly true for $|G| = 1$.

Let N be a maximal normal subgroup of G: we may suppose without loss of generality that $|G/N| = p \in \pi$. Thus V_N and V'_N are irreducible; and if $U_N = U_1 \oplus \cdots \oplus U_S$, $U'_N = U'_1 \oplus \cdots \oplus U'_t$, then by hypothesis $U_i \otimes V_N$ is irreducible and isomorphic to $U_j' \otimes V_N'$ for some j. The inductive hypothesis gives us that $V_N \cong V_N'$, $U_i \cong U_j'$, and then Lemma 2.3(ii) gives us that $V \cong V'$. We assume that $V = V'$.

If U_N is irreducible, so is U_N' (by a dimension count), and moreover so are $(U \otimes V)_N$, $(U' \otimes V)_N$. Thus $U \otimes V$, $U' \otimes V$ are irreducible. Since $U_N \cong U_N'$, we have by Huppert and Blackburn [9, Corollary 7.9.13] that $U' \cong U \otimes T$, where T is an irreducible G/N-module: we then have $T \otimes U \otimes V \cong U \otimes V$, and then Huppert and Blackburn [9, Theorem 7.9.12] tells us that T must be the trivial irreducible G/N-module. Thus $U \cong U'$ in this case. If U_N is reducible, then $U = U_i^G$, $U' = U_j'^G$, and hence $U \cong U'$. By Huppert and Blackburn [9, Lemma 7.4.15], $(U_i \otimes V_N)^G \cong U_i^G \otimes V = U \otimes V$, and then, since all the $U_j \otimes V_N$ are distinct (by the inductive hypothesis) we have by Curtis and Reiner [4, Corollary 45.5] that $U \otimes V$ is irreducible.

Now suppose that $\mathscr{P}$ is a partition of $\mathbb{P}$, the set of all primes, so that $\mathscr{P} = \{\pi_i, i \in I\}$, $\bigcup_i \pi_i = \mathbb{P}$, $\pi_i \cap \pi_j = \varnothing$ for $i \neq j$. We will say that $U \in \operatorname{Irr}_K(G)$ is $\mathscr{P}$-factorable if $U \cong U_{i_1} \otimes \cdots \otimes U_{i_n}$, where U_{i_j} is π_{i_j}-special, $\pi_{i_j} \in \mathscr{P}$, $j = 1,..., n$; unless otherwise stated, we assume that the U_{i_j} given are all non-trivial KG-modules. If $\mathscr{P} = \{\pi, \pi'\}$, we will follow Isaacs [11], and call a $\mathscr{P}$-factorable module π-factorable.

We get as an immediate Corollary to Theorem 2.4 the behaviour of $\mathscr{P}$-factorable modules on restriction to normal subgroups.

2.5. Corollary. *Let $U \in \operatorname{Irr}_K(G)$ be $\mathscr{P}$-factorable, and N a normal subgroup of G. Then U_N has all its irreducible constituents $\mathscr{P}$-factorable.*

Next we want to consider the following situation: $G = NM$, where N, M are normal subgroups of G, $U \in \operatorname{Irr}_K(G)$ with the property that U_M, U_N both have all their irreducible constituents $\mathscr{P}$-factorable. The conclusion we want is of course that U itself is $\mathscr{P}$-factorable, and for its proof we need the following result (cf. Isaacs [11, Proposition 2.7]).

2.6. Lemma. *Let $G = AB$, with A, B normal subgroups of G, and $A/A \cap B$ a π-group, $B/A \cap B$ a π'-group. Let $U \in \operatorname{Irr}_K(A \cap B)$, with $U = V \otimes W$, V π-special and invariant in B, W π'-special and invariant in A. Then every composition factor of U^G is π-factorable.*

Proof. Assume $A \cap B < G$, and work by induction on $|G/A \cap B|$. Let $L = A \cap B$, and H/L be a chief factor of G: we may assume that H/L is a π-group, so that $H \leqslant A$. Since A, BH satisfy the hypotheses of the lemma, it will be enough to show that every composition factor of U^H is π-factorable, with π-special factor invariant in BH, and π'-special factor invariant in A.

Since W is π'-special, it has a unique π'-special extension to H, $\hat{W}$ say: by the uniqueness of $\hat{W}$ and the invariance of W in A, we have $\hat{W}$ invariant in A also. We have (again by Huppert and Blackburn [9, Lemma 7.4.15]) $U^H = (V \otimes W)^H = (V \otimes \hat{W}_L)^H = V^H \otimes \hat{W}$.

By Lemma 2.3(i), every composition factor of V^H is π-special, and then by Theorem 2.4, every composition factor of U^H is π-factorable. To see that every composition factor of V^H is invariant in BH, we note that by Lemma 2.3(ii) V has unique π-special extension to B, $\hat{V}$ say, and then, by the Mackey subgroup theorem

$$\hat{V}^{BH}{}_H = \hat{V}_{B \cap H}{}^H = \hat{V}_L{}^H = V^H.$$

Thus if T is a composition factor of V^H, it is isomorphic to an irreducible constituent of S_H for some composition factor S of V^{BH}. But by Lemma 2.3(i), every composition factor of $\hat{V}^{BH}$ is π-special, and then since H has π'-index in BH, S_H is irreducible. Thus $T \cong S_H$, and is invariant in BH. This completes the proof.

We are now in a position to prove

2.7. THEOREM. *Let $G = MN$, M and N are normal subgroups of G, and let $U \in \mathrm{Irr}_K(G)$ such that U_M, U_N have all their irreducible constituents $\mathscr{P}$-factorable. Then U is $\mathscr{P}$-factorable.*

Proof. The proof is by induction on $|G|$: the result is clearly true for $|G| = 1$. Note that every irreducible G-module is $\mathscr{P}$-factorable if G is nilpotent.

We show it is enough to prove the result for M, N maximal normal subgroups of G. If M_0, N_0 are maximal normal subgroup of G with $M \leqslant M_0$, $N \leqslant N_0$, then $G = M_0 N_0$, $M_0 = M(M_0 \cap N)$, $N_0 = N(N_0 \cap M)$. By Corollary 2.5, every irreducible constituent of $U_{M_0 \cap N}$, $U_{N_0 \cap M}$ is $\mathscr{P}$-factorable, and so (by our inductive hypothesis) every irreducible constituent of U_{M_0}, U_{N_0} is $\mathscr{P}$-factorable. Thus we assume that M, N are maximal, with $|G/M| = p$, $|G/N| = r$.

Put $P = M \cap N$, and let W be an irreducible constituent of U_p. Then by assumption W is $\mathscr{P}$-factorable, say $W = W_1 \otimes \cdots \otimes W_n$, where W_j is π_{i_j}-special, $j = 1, \ldots, n$.

Suppose first that neither p nor r is contained in π_{i_j} for some j; we may as well suppose $j = 1$. Let X, Y be irreducible constituents of U_M, U_N respectively with $W \leqslant X_P$, $W \leqslant Y_P$. Since X and Y are $\mathscr{P}$-factorable, we have $\mathscr{P}$-factorizations $X = X_1 \otimes \cdots \otimes X_r$, $Y = Y_1 \otimes \cdots \otimes Y_s$, and we suppose X_1, Y_1 are the π_{i_1}-factors of X and Y respectively. Then X_{1P}, Y_{1P} are irreducible, and hence (by Theorem 2.4), are the π_{i_1}-factors of the irreducible constituents of X_P, Y_P respectively: in particular, we have $X_{1P} \cong Y_{1P} \cong W_1$. It follows immediately that W_1 is invariant in M and in N, and hence in G. Since G/P is a π'_{i_1}-group, it follows from Lemma 2.6

(with $W_1 = V$, $W_2 \otimes \cdots \otimes W_n = W$, $A = P$, $B = G$) that U is π_{i_1}-factorable: say $U = S \otimes T$, where T is π_{i_1}-special. We get immediately that T must be the unique π_{i_1}-special extension of W to G. Moreover $(S \otimes T)_M = S_M \otimes T_M$ has every irreducible constituent $\mathscr{P}$-factorable. Since T_M is π_{i_1}-special, it follows that every irreducible constituent of S_M is $\mathscr{P}$-factorable; and similarly every irreducible constituent of S_N is $\mathscr{P}$-factorable. Further for some irreducible constituent W_0 of S_P, we have $W_0 \cong W_2 \otimes \cdots \otimes W_n$.

Thus it will be enough to prove the result when either p or $q \in \pi_{i_j}$ for each π_{i_j}, $j = 1,\ldots, n$.

Suppose first that p and r are both in π_{i_1}. Then W is π_{i_1}-special, and G/P is a π_{i_1}-group. It follows from Lemma 2.3(i) that U is π_{i_1}-special (and so, in particular, $\mathscr{P}$-factorable).

Next suppose that $p \neq r$, and $p \in \pi_{i_1}$, $r \in \pi_{i_2}$, $W = W_1 \otimes W_2$ where W_1 is π_{i_1}-special, W_2 is π_{i_2}-special. Then, with $X = X_1 \otimes \cdots \otimes X_r$ as above, we have $X_{1P} \cong W_1$, and X_i is trivial (and hence so is X_j) unless X_j is π_{i_2}-special. Thus $X = X_1 \otimes X_2$, X_2 π_{i_2}-special. If X_2 is invariant in G, it has a unique extension $\hat{X}_2$ to G. Then we have $X^G = (X_1 \otimes \hat{X}_{2_M})^G \cong X_1^G \otimes \hat{X}_2$. But every composition factor of X_1^G is π_{i_1}-special, and hence U is $\mathscr{P}$-factorable.

Now suppose that X_2 is not invariant in G: by symmetry we may suppose also that $Y = Y_1 \otimes Y_2$, with Y_1 π_{i_1}-special, Y_2 π_{i_2}-special, and Y_1 not invariant in G. It follows that $U = Y^G = X^G$, and then by the Mackey subgroup theorem $U_P = X_P^G = \bigoplus_{x \in T} X_P$, where T is a transversal for M in G. Thus the number of irreducible constituents of U_P is divisible by p: similarly, it is divisible by r, and hence $pr \dim W$ divides $\dim U$. On the other hand U is a composition factor of W^G, and so $\dim U \leqslant pr \dim W$. Thus $\dim U = pr \dim W$, giving $U = W^G$. Now consider $X_1^G \otimes Y_2^G$: we have (using Huppert [8, Satz 5.16.6])

$$\begin{aligned}\operatorname{Hom}_{KG}(U, X_1^G \otimes Y_2^G) &= \operatorname{Hom}_{KG}(W^G, X_1^G \otimes Y_2^G)\\ &\cong \operatorname{Hom}_{KP}(W, (X_1^G \otimes Y_2^G)_P)\\ &= \bigoplus_{x,y} \operatorname{Hom}_{KP}(W, W_1^x \otimes W_2^y)\end{aligned}$$

where x ranges over a set of coset representatives for M in G, and y ranges over a set of coset representatives for N in G. Thus

$$\operatorname{Hom}_{KG}(U, X_1^G \otimes Y_2^G) \neq 0,$$

and so U is isomorphic to a composition factor of $X_1^G \otimes Y_2^G$. But by Lemma 2.3(i), every composition factor of X_1^G is π_{i_1}-special, and every composition factor of Y_2^G is π_{i_2}-special, and so every composition factor of $X_1^G \otimes Y_2^G$ is $\mathscr{P}$-factorable. Thus U is $\mathscr{P}$-factorable, completing the proof.

3. A Construction for Fitting Formations

Hawkes first constructed non-saturated Fitting formations by imposing conditions on the modules that could occur as chief factors of groups in the class. We will formalize his construction in the following way: if we have for each group a class of modules over a field of characteristic q specified, and the family of these classes satisfies certain closure operations, we can use this family to define a Fitting formation, which we can describe roughly as the class of all groups whose q-chief factors come from the specified class. We follow the development given in Kanes [12]: the closure properties we use (M1–M5 below) were chosen because they make the construction work, and it may be possible to choose a better set.

Let K be a field of characteristic q (not necessarily algebraically closed). Suppose that for each group G, we are given a class of modules $M(G) \subseteq \mathrm{Irr}_K(G)$: let $\mathcal{M}$ be the family of all $M(G)$. We will call $\mathcal{M}$ a Fitting family if it satisfies the following closure properties.

M1. The trivial irreducible KG-module is in $M(G)$.

M2. If $V \in M(G)$, $N \lhd G$, with $N \subseteq C_G(V)$, then V (regarded in the natural way as a G/N-module) is in $M(G/N)$.

M3. If $V \in M(H)$, and ϕ: $G \to H$ is an epimorphism, then V (regarded in the natural way as a G-module) is in $M(G)$.

M4. If $V \in M(G)$, $N \lhd G$, and U an irreducible constituent of V_N, then $U \in M(N)$.

M5. If $V \in \mathrm{Irr}_K(G)$, $G = N_1 N_2$, N_1, N_2 normal subgroups of G, and if for each irreducible constituent U of V_{N_i}, we have $U \in M(N_i)$, $i = 1, 2$, then $V \in M(G)$.

Now, following Hawkes [7], if H/L is a q-chief factor of G, we can regard H/L as a $GF(q)G$-module (by conjugation), and so define a class $\Gamma_K(G)$ of KG-modules by

$$\Gamma_K(G) = \{U \in \mathrm{Irr}_K(G)\colon U \text{ is isomorphic to a composition factor of } (H/L) \otimes_{GF(q)} K,\ H/L \text{ a } q \text{ chief factor of } G\}.$$

We now define a class of groups $\mathbf{M}$ by setting

$$\mathbf{M} = \{G\colon \Gamma_K(G) \subseteq M(G)\}.$$

The next theorem is the main result of this section.

3.1. Theorem. *If $\mathcal{M}$ is a Fitting family,* $\mathbf{M}$ *is a Fitting formation.*

Proof. The proof is quite straightforward, and is basically the proof used by Hawkes in [7].

That $\mathbf{M}$ is a class of groups follows from M3: if G is isomorphic to H, and $\Gamma_K(G) \subseteq M(G)$, then $\Gamma_K(H) \subseteq M(H)$.

Since $\mathbf{M}$ is defined by a condition on chief factors, $\mathbf{M}$ is a formation.

Now suppose that N is a normal subgroup of G. We may choose a chief series for G that includes N: then every chief factor of N is isomorphic (by the Jordan–Hölder theorem) to an irreducible constituent of the restriction to N of some chief factor of G contained in N. The commutativity of extension of the field and restiction, together with M4, gives $\Gamma_K(N) \subseteq M(N)$ if $\Gamma_K(G) \subseteq M(G)$.

To see that $\mathbf{M}$ is normal product closed, suppose that $G = N_1 N_2$, with N_1, N_2 normal subgroups of G, $N_1 \in \mathbf{M}$, $N_2 \in \mathbf{M}$, and choose a chief series for G passing through N_1 and $N_1 \cap N_2$. Let $H/L = U$ be a chief factor in this series. If $L \supseteq N_1$, we may regard U in a natural way as a G/N_1-module, and hence as an $N_2/N_2 \cap N_1$-module: as such, the irreducible constituents of U^K lie in $M(N_2/N_2 \cap N_1)$ by M2, and hence in $M(G)$ by M3. A similar argument gives the irreducible constituents of U^K in $M(G)$ if $L \supseteq N_1 \cap N_2$. Hence we suppose that $H \subseteq N_1 \cap N_2$. Then we have $U^K_{N_i}$ has all its irreducible constituents in $M(N_i)$, $i = 1, 2$, by assumption: it then follows from M5 that U^K has all its irreducible constituents in $M(G)$. Thus $\Gamma_K(G) \subseteq M(G)$, and so $G \in \mathbf{M}$, as required.

We are of course interested in using Theorem 3.1 to construct non-saturated Fitting formations. This will depend in general on the choice of $\mathscr{M}$. It is worth noting that the theorem places no restriction on the field K. The examples we give here all require K to be algebraically closed. We expect that examples can be constructed along the lines of those given in the next section without requiring the algebraic closure of K. As we have already observed, the problems lie in lack of an adequate understanding of the representation theory involved.

4. Examples

Let K be an algebraically closed field of characteristic q, $\mathscr{P}$ a partition of the set $\mathbb{P}$ of primes: so that $\mathscr{P} = \{\pi_i : i \in I\}$, $\bigcup_{i \in I} \pi_i = \mathbb{P}$, $\pi_i \cap \pi_j = \varnothing$ if $i \neq j$ (we may suppose for convenience that $\mathscr{P}$ has been labelled so that $q \in \pi_1$). For $i \in I$, let $\mathbf{X}_i$ be a Fitting formation. We now define for each group G a class of modules as follows:

$$M(G) = \{M \in \mathrm{Irr}_K(G)\colon M \text{ is } \mathscr{P}\text{-factorable, } M = M_1 \otimes \cdots \otimes M_n,$$
$$\text{with } M_j\ \pi_{i_j}\text{-special, and } G/C_G(M_j) \in \mathbf{X}_{i_j}, j = 1, \ldots, n\}.$$

We now set $\mathscr{X} = \{\mathbf{X}_i : i \in I\}$, and let

$$\mathbf{M}(q, \mathscr{P}, \mathscr{X}) = \{G\colon \Gamma_K(G) \subseteq M(G)\}.$$

4.1. THEOREM. $\mathbf{M}(q, \mathscr{P}, \mathscr{X})$ *is a Fitting formation.*

Proof. By Theorem 3.1, we need only show that the family of $M(G)$ is a Fitting family.

The properties M1, M2, M3 are clearly satisfied. If $V \in M(G)$, and N is a normal subgroup of G, then every irreducible constituent U of V_N is $\mathscr{P}$-factorable by Corollary 2.5; and if $U = U_1 \otimes \cdots \otimes U_n$, with U_j π_{i_j}-special, U_j is an irreducible constituent of the restriction of the π_{i_j}-factor V_j of V, and hence $C_N(U_j) \geqslant C_G(V_j) \cap N$, whence $N/C_N(U_j) \in \mathbf{X}_{i_j}$. Thus M4 holds.

Now suppose that $G = AB$, with A, B normal in G, $V \in \mathrm{Irr}_K(G)$, with each irreducible constituent of V_A, V_B in $M(A)$, $M(B)$ respectively. We have immediately from Theorem 2.7 that V is $\mathscr{P}$-factorable: say $V = V_1 \otimes \cdots \otimes V_n$, where V_j is π_{i_j}-special. Suppose that $(V_j)_A = X_{j1} \oplus \cdots \oplus X_{jr}$, $(V_j)_B = Y_{j1} \oplus \cdots \oplus Y_{js}$, where the X_{jk}, Y_{jk} are irreducible. Then

$$C_A(V_j) = \bigcap_k C_A(X_{jk}), \qquad C_B(V_j) = \bigcap_k C_B(Y_{jk}).$$

Thus $A/C_A(V_j) \in \mathbf{X}_{i_j}$, $B/C_B(V_j) \in \mathbf{X}_{i_j}$. Also $C_A(V_j)\, C_B(V_j) \leqslant C_G(V_j)$, and $G/(C_A(V_j)\, C_B(V_j))$ is the normal product of $AC_B(V_j)/C_A(V_j)\, C_B(V_j)$ and $BC_A(V_j)/C_A(V_j)\, C_B(V_j)$. But $AC_B(V_j)/C_A(V_j)\, C_B(V_j) \cong A/C_A(V_j) \in \mathbf{X}_{i_j}$, $BC_A(V_j)/C_A(V_j)\, C_B(V_j) \cong B/C_B(V_j)$, and so $G/C_G(V_j) \in QN_0\mathbf{X}_{i_j} = \mathbf{X}_{i_j}$. This establishes that M5 holds.

Since our main interest is in constructing non-saturated Fitting formations, we want to be able to distinguish the non-saturated ones among the $\mathbf{M}(q, \mathscr{P}, \mathscr{X})$. Though it is easy to produce non-saturated examples by appropriate choices of $\mathscr{P}$ and $\mathscr{X}$, a characterization of the non-saturated ones is more difficult.

We take $\mathbf{M} = M(q, \mathscr{P}, \mathscr{X})$ as above, and set $\mathbf{M}_1 = M(q, \mathscr{P}, \mathscr{Y})$, where $\mathscr{Y} = \{\mathbf{Y}_i : i \in I\}$, with $\mathbf{Y}_1 = \mathbf{S}$, and $\mathbf{Y}_i = \mathbf{X}_i$ for $i > 1$. Note that $\mathbf{M} \subseteq \mathbf{M}_1$. We can now give criteria for the saturation of $\mathbf{M}$.

4.2. THEOREM (L. G. Kovács). *The following conditions are equivalent*:

(i) $\mathbf{M}$ *is saturated,*

(ii) $\mathbf{M} = \mathbf{S}_{q'}\mathbf{S}_q[\bigvee_{i \in I} (\mathbf{X}_i \cap \mathbf{S}_{\pi_i})]$ (where $\bigvee_{i \in I} (\mathbf{X}_i \cap \mathbf{S}_{\pi_i})$ *denotes the Fitting formation generated by the* $\mathbf{X}_i \cap \mathbf{S}_{\pi_i}$),

(iii) $\mathbf{X}_1 \cap \mathbf{M}_1 \subseteq \mathbf{S}_{\pi_1' \cup \{q\}}\, \mathbf{S}_{\pi_1}$, *and for* $i > 1$, $\mathbf{X}_i \subseteq \mathbf{S}_{\pi_i'}\mathbf{S}_{\pi_i}$.

It is worth making a couple of remarks before we prove Theorem 4.2. First, when $\mathbf{M}$ really is a "new" Fitting formation (rather than an "old" one, more easily described as a product, and by its form clearly saturated) it is never saturated. Second, the saturation of M depends not only on the individual properties of the $\mathbf{X}_i$ but also on their collective interaction. The nature of that interaction is fairly clear from (iii): we give an example to make the interaction more explicit.

The example we give in the following paragraph, as well as all the examples of the next section, use partitions of the set of primes into two subsets: we will introduce a different notation for this case which makes the examples easier to describe. Thus suppose $\mathscr{P} = \{\pi_1, \pi_2\}$; we put $\pi_1 = \pi$, so that $\pi_2 = \pi'$. If $\mathscr{X} = \{\mathbf{X}_1, \mathbf{X}_2\}$ we set

$$\mathbf{M}_q^\pi(\mathbf{X}_1, \mathbf{X}_2) = \mathbf{M}(q, \mathscr{P}, \mathscr{X}).$$

Now put $\pi = \{q, p\}$, where p is a prime different to q, and set $\mathbf{Y} = \mathbf{M}_p^\pi(\mathbf{S}, \mathbf{E})$ (where $\mathbf{E}$ is the class of groups of order 1). We will show that $\mathbf{M}_q^\pi(\mathbf{Y}, \mathbf{E})$ and $\mathbf{M}_q^\pi(\mathbf{E}, \mathbf{S}_\pi \mathbf{S}_{\pi'})$ are saturated, but $\mathbf{M}_q^\pi(\mathbf{Y}, \mathbf{S}_\pi \mathbf{S}_{\pi'})$ is not saturated. That $\mathbf{M}_q^\pi(\mathbf{E}, \mathbf{S}_\pi \mathbf{S}_{\pi'})$ satisfies (iii) is immediate, and its saturation follows. To see that $\mathbf{M}_q^\pi(\mathbf{Y}, \mathbf{E})$ satisfies (iii), the only non-trivial step is to show that $\mathbf{Y} \cap \mathbf{M}_q^\pi(\mathbf{S}, \mathbf{E}) \subseteq \mathbf{S}_{\pi' \cup \{q\}} \mathbf{S}_\pi$. Suppose not, and that G is minimal with $G \in \mathbf{Y} \cap \mathbf{M}_q^\pi(\mathbf{S}, \mathbf{E})$, $G \notin \mathbf{S}_{\pi' \cup \{q\}} \mathbf{S}_\pi$. We may assume that G has a unique maximal normal subgroup N such that $|G/N| = r \notin \pi$, and that $O_{\pi' \cup \{q\}}(G) = 1$. Thus we have $N \in \mathbf{S}_\pi$. If N/M is a chief factor of G, it is either a p-group or a q-group on which G/N acts non-trivially. In the first case we get $G/N \notin \mathbf{Y}$, in the second case $G/N \notin M_q^\pi(\mathbf{S}, \mathbf{E})$: in either case, a contradiction. Thus $\mathbf{M}_q^\pi(\mathbf{Y}, \mathbf{E})$ is saturated. Now let $r \in \pi'$, and let Q be an extraspecial group, chosen minimal such that a group R of order r acts non-trivially on Q but trivially on Q'. Let P be a faithful irreducible $GF(p)$ QR-module, and form the split extension $H = PQR$. Then we have $H \in \mathbf{Y}$, $H \notin \mathbf{S}_{p'} \mathbf{S}_\pi$, but $H \in \mathbf{M}_q^\pi(\mathbf{S}, \mathbf{S}_\pi \mathbf{S}_{\pi'})$: thus $\mathbf{M}(\mathbf{Y}, \mathbf{S}_\pi \mathbf{S}_{\pi'})$ does not satisfy (iii), and so is not saturated.

Proof (*of Theorem* 4.2). If U_i is a π_i-special module, then $O_{\pi_i'}(G/C_G(U_i)) = 1$. Also, since U_i is irreducible in characteristic q, $O_q(G/C_G(U_i)) = 1$. Thus $O_{\pi_i' \cup \{q\}}(G/C_G(U_i)) = 1$. If moreover $G/C_G(U_i) \in \mathbf{X}_i \cap \mathbf{M} \subseteq \mathbf{X}_i \cap \mathbf{M}_1 \subseteq \mathbf{S}_{\pi_i' \cup \{q\}} \mathbf{S}_{\pi_i}$, then $G/C_G(U_i) \in \mathbf{X}_i \cap \mathbf{S}_{\pi_i}$, and so (iii) implies that $\mathbf{M} \subseteq \mathbf{S}_{q'} \mathbf{S}_q [\bigvee_{i \in I} (\mathbf{X}_i \cap \mathbf{S}_{\pi_i})]$. Since the converse inclusion always holds, we have that (iii) implies (ii). That (ii) implies (i) follows immediately from the form of $\mathbf{M}$. It remains to show that (i) implies (iii). We suppose it does not, and show that this leads to a contradiction.

To simplify our notation, we put $\pi = \pi_j$ and $\tau = \pi' \cup \{q\}$ (so that $\tau = \pi'$ unless $j = 1$).

Suppose first that $j = 1$, so that $\mathbf{M}_1 \cap \mathbf{X} \nsubseteq \mathbf{S}_\tau \mathbf{S}_\pi$, and H is a group of least order showing this. Such an H clearly has a unique maximal normal subgroup, M say, and a unique minimal normal subgroup, P say. Since $\mathbf{S}_\tau \mathbf{S}_\pi$ is saturated, H must have trivial Frattini subgroup, and so P is complemented; let L be a complement, and note $C_L(P) = 1$. Since $L \in \mathbf{S}_\tau \mathbf{S}_\pi$, we must have that $|P| = p^k$ for some prime p with $q \neq p \in \pi$. Thus we have

$O_\tau(H) = O_\tau(M) = 1$, and so M, being in $\mathbf{S}_\tau \mathbf{S}_\pi$, is a π-group. Since $H \notin \mathbf{S}_\pi$, the prime index, say r, of M in H must lie in π' and hence in π_i for some $i \neq 1$. Since H can have no non-trivial π-quotient, we must have $H/P \in \mathbf{S}_\tau$. It follows that $M/P \in \mathbf{S}_\pi \cap \mathbf{S}_\tau = \mathbf{S}_q$. If $M/P = 1$, then clearly $H \in \mathbf{M}$. If $M/P \neq 1$, then any q-chief factor of H is either central or has centralizer M: in either case the irreducible KH-modules coming from it are π_i-special. Since $H \in \mathbf{M}_1$ and $i \neq 1$, we must have $H/M \in \mathbf{X}_i$. Thus $H \in \mathbf{M}$.

If $j \neq 1$, and H is a group of least order in $\mathbf{X}_j$ but not in $\mathbf{S}_\tau \mathbf{S}_\pi$, the above analysis gives $H = PL$, where P is a p-group for some $p \in \pi$, and L is an r-group for some $r \in \pi'$. Since $q \notin \pi$, $H \in \mathbf{S}_{q'} \mathbf{S}_q \subseteq \mathbf{M}$.

Our next step is to construct two groups, G_1 and G_2 say, with the following properties. First, $G_1 \in \mathbf{X}_j \cap \mathbf{M} \cap \mathbf{S}_\pi \mathbf{S}_{\pi'}$, with $O_{\pi'}(G_1) = 1$, and G_2 lying in the formation generated by G_1. Second, there exists a faithful irreducible $KO_\pi(G_1)$-module W_1 which is G_1-invariant. Third, there exists an irreducible $KO_\pi(G_2)$-module W_2 which is not G_2-invariant. The groups G_1 and G_2 are derived from the structure of the group H above.

If $M/P = 1$, we set $G_2 = H$, and let W_2 be any non-trivial irreducible KP-module. Let P_1 be an extra special group of order p^{2k+1}, and let R be a group of order r acting on P_1 so that R acts on the Frattini quotient of P_1 as L acts on the direct sum of P and its contragredient, and trivially on the centre of P_1 (see Huppert [8, Hilfssatz VI.7.22] for the construction of such a group). Set $G_1 = P_1 R$. The Frattini quotient of G_1 is a subdirect square of H, and so lies in the metanilpotent Fitting formation $\mathbf{X}_j \cap \mathbf{M} \cap \mathbf{S}_p \mathbf{S}_r$: all such formations are saturated (Hawkes [7, Theorem 1]) and so $G_1 \in \mathbf{X}_j \cap \mathbf{M}$. Moreover, G_2 is a quotient of G_1 and so is in the formation it generates. Finally, faithful irreducible KP_1 modules are invariant under automorphisms of P_1 which fix its centre (Huppert [8, V.16.14]): hence any such KP_1-module will serve as W_1.

It is somewhat harder to deal with the case $M/P \neq 1$. In this case $O_\pi(H) = M$, and $r \neq q$. Put $Q = M \cap L$, and let R be a Sylow r-subgroup of L. As $PQ = M$, the unique maximal normal subgroup of H, we must have $[R, Q] = Q$.

If $[P, R] \neq P$, set $G_1 = H$. Let P_0 be a maximal subgroup of P containing $[P, R]$, and let Q_0 be the largest subgroup of Q which normalizes P_0, and hence also Q_0, is normalized by R, and also that $P_0 Q_0$ is normal in PQ_0 (of index p). Let W_1 be the KM-module induced from any one dimensional KPQ_0-module W with $C_{PQ_0}(W) = P_0 Q_0$. The choice of Q_0 ensures that PQ_0 is the inertia subgroup in M of the restriction of W_1 to P, and hence W_1 is irreducible. As R acts trivially on $PQ_0/P_0 Q_0$ we know that W is R-invariant, and hence so is W_1. It follows that $C_M(W_1)$ is normal in G_1: as it does not contain P, it must be trivial.

If $[P, R] = P$, we construct G_1 as a group which is like H in every relevant respect except this. (Of course, G_1 will not retain the minimal property of H either. The construction makes no use of $[P, R]=P$, and could be performed in any case: the only reason for handling $[P, R]\neq P$ separately was to show how this property is exploited before getting submerged in other complications.). Let $R=\langle h\rangle$, and set $h_0=(h^{-1}, h)\in QR\times QR$, $Q_0=Q\times Q\leqslant QR\times QR$, $R_0=\langle h_0\rangle$. Then Q_0R_0 is normal in $QR\times QR$, and $[Q_0, R_0]=Q_0$. Let P_1 and P_2 be $(QR\times QR)$-modules such that on P_1 the first factor acts as on P and the second factor acts trivially, and on P_2 the second factor acts as on P and the first factor acts trivially. Consider $P_1\otimes P_2$: as $QR\times 1$ or $1\times QR$ module, $P_1\otimes P_2$ is a direct sum of "isomorphic copies" of P. It follows from $C_P(Q)=1$ that $C_{P_1\otimes P_2}(Q_0)=1$. On the other hand, $(P_1\otimes P_2)_{R_0}$ is just the tensor product of P_{1R_0} and its contragredient, and so $C_{P_1\otimes P_2}(R_0)\neq 1$. Now consider the semidirect product $(P_1\otimes P_2)(QR\times QR)$. This is the normal product of $(P_1\otimes P_2)(QR\times 1)$ and $(P_1\otimes P_2)(1\times QR)$, each of which is a subdirect power of $PQR=H$, and so lies in the Fitting formation $\mathbf{X}_1\cap\mathbf{M}\cap\mathbf{S}_\pi\mathbf{S}_{\pi'}$. Consequently so does its normal subgroup $(P_1\otimes P_2)\,Q_0R_0$. By Maschke's theorem, $P_1\otimes P_2$ is completely reducible as Q_0R_0-module: the observation $C_{P_1\otimes P_2}(R_0)\neq 1$ yields that $P_1\otimes P_2$ has an irreducible Q_0R_0-submodule P_0 such that $C_{P_0}(R_0)\neq 1$. Let S be a Q_0R_0-submodule complementing P_0 in $P_1\otimes P_2$, and T a normal subgroup of $(P_1\otimes P_2)\,Q_0R_0$ maximal with respect to containing S but avoiding P_0. We set $G_1=(P_1\otimes P_2)\,Q_0R_0/T$. Clearly P_0T/T is operator isomorphic to P_0, and is both the unique Sylow p-subgroup and the unique minimal normal subgroup of G_1. Since $C_{P_1\otimes P_2}(Q_0)=1$, we have $[P_0, Q_0]=P_0$, so that $Q_0\nleqslant T$. Moreover, $[Q_0, R_0]=Q_0$, and so also $R_0\nleqslant T$. On the other hand, $C_{P_0}(R_0)\neq 1$ ensures that $[P_0T/T, R_0T/T]\neq P_0T/T$. These facts guarantee that the present G_1 has all the relevant properties of the G_1 of the previous paragraph, and we may construct W_1 as above.

To simplify the description of G_2 and W_2, we change notation, and from now on P, Q, R will stand for appropriate Sylow subgroups or G_1. If P has a maximal subgroup N not containing any conjugate of $[P, R]$, we take $G_2=G_1$, and let W be a one dimensional KP-module with $C_P(W)=N$, and W_2 be any irreducible KPQ-module whose restriction to P contains W (that is, any composition factor of W^{PQ}). By its choice, W is not invariant under any conjugate of R, so that the number of isomorphism types of G_2-conjugates of W is divisible by r. If W_2 were G_2-invariant, the set of isomorphism types of irreducible submodules of W_{2P} would be G_2-invariant. But by Cliffords theorem that set is a single Q-orbit, and so the cardinality of this set would be a power of q divisible by r. As this is impossible, W_2 is not G_2-invariant, and we are done.

If P does not have such a maximal subgroup P_2, we choose G_2 differently: if $|P|=p^l$, we take for G_2 the semidirect product of QR with the l-fold direct power P^l of P. Then G_2 is a subdirect power of G_1, and so lies in the formation generated by G_1. We show that there is a maximal subgroup P_2 of P^l not containing any conjugate of $[P^l, R]$, and then construct W_2 as in the previous paragraph. We have that P itself is the l-fold direct power of a group C of order p, and we let $\phi_1, \ldots, \phi_l$ be the corresponding coordinate projections of P onto C. Note that the intersections of the kernels of these projections is trivial. Define a homomorphism ϕ from P^l to C by

$$(x_1, \ldots, x_l)\phi = \prod (x_i \phi_i);$$

and set $P_2 = \ker \phi$. Since $[Q, R] = Q$, the normal closure of R contains Q, and hence R cannot act trivially on P, otherwise Q also would, contrary to $C_Q(P) = 1$. Thus to each element g of QR there is an index $i(g)$ such that $[P, R]^g \phi_{i(g)} = 1$, and hence an element $y_g \in P$ such that $[y_j, h^g]\phi_{i(g)} \neq 1$ (where $R = \langle h \rangle$). Let z_g be the element of P^l whose components are all trivial except the $i(g)$-component, which is y_g, and so

$$[z_g, h^g]\phi = [y_g, h^g]\phi_{i(g)} \neq 1.$$

Hence $[P^l, R^g]$, that is $[P^l, R]^g$, is not contained in P_2, as required.

The final step of the proof now goes easily, as follows. By Lemma 2.3(ii), W_1 is the restriction of a unique π-special KG_1-module U_1 (which is faithful since $O_{\pi'}(G_1) = 1$); in turn U_1 is a submodule of $K \otimes V_1$ for some faithful irreducible $GF(q)$ G_1-module V_1. It is an easy check to see that $V_1 G_1$ lies in $\mathbf{M}$. On the other hand, W_2 is a submodule of the restriction of some irreducible KG_2-module U_2, and since W_2 is not G_2-invariant, that restriction cannot be homogeneous. This U_2 is not $\mathscr{P}$-factorable: indeed, it is not even π-factorable. For suppose $U_2 = U \otimes U'$, where U' is π'-special. Then the restriction of U' to $O_\pi(G_2)$ must be trivial, and so the restriction of U to $O_\pi(G_2)$ contains W_2 and is therefore inhomogeneous. The dimension of U is then divisible by the non-trivial index of the inertia subgroup of W_2 in G_2: thus this dimension cannot be a π-number, and U cannot be π-special. If V_2 is an irreducible $GF(q)$ G_2-module such that $K \otimes V_2$ contains U_2, then $V_2 G_2 \notin \mathbf{M}$. On the other hand, since our formation is saturated, there is a formation $\mathbf{F}$ such that $\mathbf{F} = \mathbf{S}_q \mathbf{F} \subseteq \mathbf{M} \subseteq \mathbf{S}_{q'} \mathbf{F}$ (see Carter and Hawkes [3]). Since V_1 is faithful, $O_{q'}(V_1 G_1) = 1$, and so $V_1 G_1 \in \mathbf{M} \subseteq \mathbf{S}_{q'} \mathbf{F}$ implies that $V_1 G_1 \in \mathbf{F}$. Since G_2 lies in the formation generated by G_1, we have $G_2 \in \mathbf{F}$, and hence $V_2 G_2 \in \mathbf{S}_q \mathbf{F} \subseteq \mathbf{M}$. This gives us the desired contradiction, and the proof of Theorem 4.2 is complete.

We should remark that the difficult cases in the construction of the groups G_1 and G_2 above do arise. Take $q = 2$, $\pi = \{2, 13\}$, and $\mathbf{X} = \mathbf{S}_{13}\mathbf{S}_2\mathbf{S}_3 \cap \mathbf{M}_{13}^3(\mathbf{E}, \mathbf{S}_{3'}\mathbf{S}_3)$: then the smallest group H in $\mathbf{X}$, but outside $\mathbf{S}_{\pi' \cup \{2\}}\mathbf{S}_\pi$ has H/P isomorphic to $SL(2, 3)$, and hence $[P, R] = P$. On the other hand, take $q = 31$, $\pi = \{5, 31\}$, and $\mathbf{X} = \mathbf{S}_5\mathbf{S}_{31}\mathbf{S}_5 \cap \mathbf{M}_5^3(\mathbf{E}, \mathbf{S}_{3'}\mathbf{S}_3)$: then we find that H/P is the nonabelian group of order 93, and each of the 31 maximal subgroups of P (which has order 5^3) is a conjugate of $[P, R]$.

5. The Known Examples

In this section we will show that the examples due to Hawkes, Berger, and Cossey, the generalizations of these given by Kanes, and the new examples of Kanes are all special cases of the examples of the previous section.

These examples all use partitions of the set of primes into two subsets: recall that if $\mathscr{P} = \{\pi_1, \pi_2\}$, we put $\pi_1 = \pi$, so that $\pi_2 = \pi'$; and if $\mathscr{X} = \{\mathbf{X}_1, \mathbf{X}_2\}$, we set

$$\mathbf{M}_q^\pi(\mathbf{X}_1, \mathbf{X}_2) = (\mathbf{M}(q, \mathscr{P}, \mathscr{X})$$

(where q is a prime, and in what follows, K is an algebraically closed field of characteristic q).

We consider the examples given in Chapter 4 of Kanes [12] first. Let $\mathbf{x}$ be a Fitting formation such that $\mathbf{X} \subseteq \mathbf{S}_{\pi_0'} S_{\pi_0}$, where $\pi_0 = \pi \setminus \{q\}$. For each G, we define $Y_q^\pi(G)$ to be the class of all $V \in \operatorname{Irr}_K(G)$ such that

(1) $\dim V$ is a π'-number,

(2) if $g \in G$, $o(g \text{ on } V)$ is a π'-number.

(3) $G/C_G(V) \in \mathbf{X}$.

Kanes shows that the family of $Y_q^\pi(G)$ is a Fitting family, and so by Theorem 3.1 he gets a Fitting formation he denotes $\mathbf{Y}_q^\pi(\mathbf{X})$.

We show that $\mathbf{Y}_q^\pi(\mathbf{X}) = \mathbf{M}_q^\pi(\mathbf{E}, \mathbf{X})$ (where $\mathbf{E}$ is the class of all groups of order 1). Since $\mathbf{M}_q^\pi(\mathbf{E}, \mathbf{X})$ is defined by the family of $M(G)$, where

$$\begin{aligned} M(G) = \{ V \in \operatorname{Irr}_K(G) &: V \text{ is } \pi\text{-factorable, and if } V = V_1 \otimes V_2, \text{ with} \\ & V_1\ \pi\text{-special and } V_2\ \pi'\text{-special}, G/C_G(V_1) \in \mathbf{E}, G/C_G(V_2) \in \mathbf{X} \} \\ = \{ V \in \operatorname{Irr}_K(G) &: V \text{ is } \pi'\text{-special}, G/C_G(V) \in \mathbf{X} \}, \end{aligned}$$

it is enough to show $M(G) = Y_q^\pi(G)$. Clearly $M(G) \subseteq Y_q^\pi(G)$. To see the other inclusion, suppose $V \in Y_q^\pi(G)$: we must show that V is π'-special, and to do this it will be enough to show that if N is a maximal normal subgroup of G, and U is an irreducible constituent of V_N, then U satisfies conditions (1), (2), and (3) above. (This is essentially the proof that $Y_q^\pi(G)$ satisfies M4.) Since $\dim U$ divides $\dim V$, (1) is clearly satisfied. Now V_N is either homogeneous, in which case it is irreducible and $U = V_N$, or $V = U^G$ (and so $|G/N| = p \in \pi'$). If $U = V_N$, then for $n \in N$, $o(n \text{ on } U) = o(n \text{ on } V)$, and so is a π'-number. Thus suppose that V_N is reducible. Since $|G/N| \in \pi'$, we must have $NO^{\pi}(G)=G$, and so we may choose $x_1=1,\ldots,x_p\in O^{\pi}(G)$ such that $V_N=U\oplus U^{x_2}\oplus\cdots\oplus U^{x_p}$. We now want to show that $o(n \text{ on } U^{x_i})$ is a π'-number. We note first that $o(n \text{ on } U^{x_i})=o(n^{x_i} \text{ on } U)$. However $\det(n^{x_i} \text{ on } U)=\det(n \text{ on } U)\det([n,x_i] \text{ on } U)$, and since $[n,x_i]\in O^{\pi}(G)$, it is a π'-element, and $o([n,x_i] \text{ on } U)$ is a π'-number. Moreover

$$\det(n \text{ on } V)=\prod_i \det(n \text{ on } U^{x_i})=(\det(n \text{ on } U))^p\prod_i \det([n,x_i] \text{ on } U).$$

Since $o(n$ on $V)$, $o([n,x_i]$ on $U)$, p are all π'-numbers, it follows that $o(n$ on $U)$ is a π'-number. This establishes (2) for N. Finally $C_N(U)\geqslant C_G(V)\cap N$, and hence $N/C_N(U)\in Q\mathbf{X}\subseteq\mathbf{X}$, establishing (3).

Note that the above proof would have worked equally well with $\mathbf{X}\subseteq\mathbf{S}_{\pi'}\mathbf{S}_{\pi}$: the reason for choosing $\mathbf{S}_{\pi 0}\mathbf{S}_{\pi 0}$ in its place was to obtain the Hawkes and Berger–Cossey examples as special cases. These come as follows.

If $\pi=\{q\}$, then $\mathbf{M}_q^q(\mathbf{E},\mathbf{S})$ is the class defined by Hawkes in [7] while if $\pi=\{p\}$, $M_q^p(\mathbf{E},\mathbf{S}_p,\mathbf{S}_p)$ is the class defined by Berger and Cossey in [2].

The next example comes from Chapter 5 of Kanes [12], and is superficially rather different from the $Y_q^{\pi}(\mathbf{X})$. Let $\mathbf{X}$ be a Fitting formation such that $\mathbf{X}\subseteq\mathbf{S}_{\pi}\mathbf{S}_{\pi'}$; for each G we define $H_q^{\pi}(\mathbf{X})$ to be the class of modules $V\in\operatorname{Irr}_K(G)$ satisfying

(i) $V_{O_{\pi}(G/C)}$ is homogeneous, where $C=C_G(V)$, and

(ii) $G/C\in\mathbf{X}$.

Again, Kanes shows that the family of $H_q^{\pi}(\mathbf{X})$ is a Fitting family and gets a Fitting formation he denotes by $\mathbf{H}_q^{\pi}(\mathbf{X})$.

We will show that $\mathbf{H}_q^{\pi}(\mathbf{X})=\mathbf{M}_q^{\pi}(\mathbf{X},\mathbf{X})$: again by showing that $H_q^{\pi}(G)=M(G)$, where

$$\begin{aligned} M(G)=\{V\in\operatorname{Irr}_K(G)\colon\ & V \text{ is } \pi\text{-factorable},\quad V=V_1\otimes V_2,\quad V_1\ \pi\text{-special},\\ & V_2\ \pi'\text{-special},\quad G/C_G(V_1),\, G/C_G(V_2)\in\mathbf{X}\}.\end{aligned}$$

First, suppose $V\in M(G)$, and put $H/C=O_{\pi}(G/C)$, $C=C_G(V)$. If $V=V_1\otimes V_2$, $V_1\pi$-special, V_2 π'-special, then $H\subseteq C_G(V_2)$, since a normal π-subgroup acts trivially on a π'-special module. Thus $(V_2)_H$ is a trivial H-module. On the other hand, V_1 is π-special and G/H is a π'-group, so that $(V_1)_H$ is irreducible. Thus V_H is a direct sum of modules isomorphic to $(V_1)_H$, and so is homogeneous. Moreover, $C_G(V)\supseteq C_G(V_1)\cap C_G(V_2)$, and hence $G/C\in QR_0(\mathbf{X})=\mathbf{X}$. Thus $V\in H_q^{\pi}(G)$ and so $M(G)\subseteq H_q^{\pi}(G)$.

In the other direction, let $V\in H_q^{\pi}(G)$. Then V_H is homogeneous by assumption, and moreover, each irreducible constituent U of V_H is π-special (since H/C is a π-group). It now follows from Lemma 2.6 that V is π-factorable: say $V=V_1\otimes V_2$, with $V_{1H}\cong U$, V_{2H} a trivial H-module. Thus

$C \subseteq C_G(V_2)$, giving $G/C_G(V_2) \in \mathbf{X}$, and $C \subseteq C_G(V_1)$, giving $G/C_G(V_1) \in \mathbf{X}$. Thus $V \in M(G)$, and so $H_q^\pi(G) \subseteq M(G)$. This completes the proof.

REFERENCES

1. H. E. BECKER, Fortsetzungen irreduzibler Darstellungen über beliebigen Körpern, *Arch. Math.* (*Basel*) **27** (1976), 588–592.
2. T. R. BERGER AND J. COSSEY, More Fitting formations. *J. Algebra*, **51** (1978), 573–578.
3. R. CARTER AND T. HAWKES, The $\mathscr{F}$-normalizers of a finite soluble group, *J. Algebra* **5** (1967), 175–202.
4. C. W. CURTIS AND I. REINER, "Representation Theory of Finite Groups and Associative Algebras," Wiley–Interscience, New York/London, 1962.
5. D. GAJENDRAGADKAR, A characteristic class of characters of finite π-separable groups. *J. Algebra* **59** (1979), 237–259.
6. W. GASCHÜTZ, "Lectures on Subgroups of Sylow Type in Finite Soluble Groups," Notes on Pure Mathematics, Vol. 11, ANU Press, Canberra, 1979.
7. T. O. HAWKES, On Fitting formations, *Math. Z.* **117** (1970), 177–182.
8. B. HUPPERT, "Endliche Gruppen I," Die Grundlehren der Mathematischen Wissenschaften 137, Springer-Verlag, Berlin/Heidelberg/New York, 1967.
9. B. HUPPERT AND N. BLACKBURN, "Finite Groups II," Die Grundlehren der Mathematischen Wissenschaften 242, Springer-Verlag, Berlin/Heidelberg/New York, 1982.
10. I. M. ISAACS, "Character Theory of Finite Groups," Academic Press, New York/San Francisco/London, 1976.
11. I. M. ISAACS, Characters of π-separable groups, *J. Algebra* **86** (1984), 98–128.
12. C. L. KANES, "Constructions for Fitting Formations," Ph. D. thesis, Australian National University, Canberra, 1982.
13. L. G. KOVÁCS, "Some Fitting Formations of Finite Soluble Groups," Mathematics Research Report 37, Australia National Univ., Canberra, 1982.